MATH 207
TEST II

I. Define the terms

A. Basis of a vector space
B. Dimension of a vector space
C. Linear transformation
D. Null space of a linear transformation
E. Range of a linear transformation

II. Computations

A. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 2 & 5 & 3 & 1 \end{pmatrix}$.

(Provide at least some justification for your answers.)
i) What is the rank of A ?
ii) What is the dimension of R(A), the row space of A?
iii) Find a basis for the row space of A.
iv) Extend your basis in (iii) to a basis for the whole vector space.
v) What is the column rank of A ?
vi) Consider multiplication by A as a linear transformation from R^4 to R^3 (i.e. Ax). Find a basis for the null space of this linear transformation.

B. Let P_1 be the set of polynomials of degree 1 or less and let T be the set of 2×2 upper triangular matrices. Consider the function $S : P_1 \to T$ defined by $S(a_0 + a_1 x) = \begin{pmatrix} a_0 & a_0 - a_1 \\ 0 & 2a_1 \end{pmatrix}$

i) Show that S is a linear transformation.
ii) Find a matrix representation for S with respect to the bases:
X= $\{1 + x, 1 - x\}$ and Y= $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
iii) Find the null space of S.
iv) What is the dimension of the range of S? Justify. (Hint: use a theorem.)

III. Theorems

A. Prove that the null space of a linear transformation $T : V \to W$ is a subspace. (Is this a subspace of V or W?)

B. Let A be a nonsingular $n \times n$ matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$