

## I. Definitions &amp; Statements of Theorems. (3 pts each)

## A.1. Subspace theorem

2. Linear independence
3. Span of a set of vectors  $\{v_1, v_2, \dots, v_n\}$
4. Basis
5. Linear transformation
6. Range of a linear transformation

## B. Theorems

1. State any theorem from the book that concerns “dimension” of a vector space.
2. State any theorem from the book that concerns linear independence.

## II. Computations.

A. Consider the matrix  $A = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 2 & 5 & 3 & 1 \\ 3 & 7 & 4 & 1 \end{pmatrix}$ .

(Provide at least some justification for your answers.)

- i) (2) What is the column rank of  $A$ ?
- ii) (2) What is the dimension of the column space of  $A$ ?
- iii) (2) Find a basis for  $\mathcal{C}(A)$ , the column space of  $A$ .
- iv) (4) Extend your basis in (iii) to a basis for the whole vector space. (What is the “whole vector space” that is referred to here?)
- v) (2) What is the row rank of  $A$ ?
- vi) (2) Find the row-column canonical form for  $A$ . (You should not need to do any extra computations to find this.)
- vii) (4) Discuss the possible solutions to  $Ax = b$ . Does a solution exist? Is it unique? Explain.

B. Let  $M_2$  be the set of all  $2 \times 2$  matrices and let  $\mathcal{P}_2$  be the set of polynomials of degree less than 2. Consider the function  $T : M_2 \rightarrow \mathcal{P}_2$  defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d) + (b - 2c)x$$

- i) (4) Show that  $T$  is a linear transformation.
- ii) (4) Find a matrix representation for  $T$  with respect to the bases:  $\mathcal{X} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$  and  $\mathcal{Y} = \{1, x\}$ .
- iii) (4) Find the null space of  $T$ . (If easier, you may describe the null space by finding a basis for it.)
- iv) (3) What is the dimension of the range of  $T$ ? Justify.

v) (3) Find the first column of the representation  $R_{\mathcal{W}\mathcal{X}}T$  of the transformation  $T$  with respect to the bases  $\mathcal{X}$  and the new basis  $\mathcal{W} = \{1 + x, 1 - x\}$ .

III. Theorems. Do TWO out of THREE. (10 points each)

A. Show that the null space of a linear transformation  $T : V \rightarrow W$  is a subspace. (Be sure to tell which space,  $V$  or  $W$ , contains this subspace.)

B. Definition: a function  $f : X \rightarrow Y$  is called one-to-one in case  $f(a) = f(b)$  implies that  $a = b$ . (This means that distinct elements in the domain go to distinct elements in the range.)

Let  $T : V \rightarrow W$  be a linear transformation. Prove that if the null space of  $T$  is  $\{0\}$ , then  $T$  is one-to-one.

C. Let  $C$  and  $D$  be subsets of a vector space  $V$ . Show that  $\text{span}(C \cap D)$  is a subspace of  $(\text{span}(C)) \cap (\text{span}(D))$ .

Extra credit: (5) (an extension of part B) Show that if  $T$  is one-to-one then the null space of  $T$  is  $\{0\}$ .