Predicting time-varying long-run variance – Modified component GARCH model approach

Jang Hyung Cho*, Assistant Professor of Finance
San Jose State University

Ahmed F. Elshahat, Assistant Professor of Finance
Bradley University

ABSTRACT

Component GARCH models have been widely used in academia. Among the component models, Engle and Lee’s (1999) two component GARCH model set the stage in the literature. However, despite the significance of Engle and Lee’s two component GARCH model, we find their long run variance component is indistinguishable from the total variance in many cases. We identify the two main conditions of coefficients of the model under which the long-run variance component is not filtered from the total conditional variance in the Engle and Lee’s model. In this paper we develop a modified component GARCH model that overcomes the aforementioned mal-filtering problem. The sample data used are from the Real Estate Investment Trust (REIT) indexes of U.S. and Japan from year 2000 to 2009. We employ the power transfer function and the squared coherence spectrum to examine the filtering performance. The results confirm that the proposed modified component GARCH outperforms the Engle and Lee’s model in filtering the time-varying long-run variance.

JEL classification: G17, C51
Keywords: component GARCH; long-run variance; filtering; spectral analysis

INTRODUCTION

Heterogeneous information arrivals impart differing volatility dynamics with differing frequencies in their impact on the market prices. Therefore, the market volatility aggregates numerous independent volatility components (Andersen and Bollerslev, 1997b). Also, the traders with different trading holding periods can lead to differing volatility components (Muller, Dacorogna, Dave, Olsen, Pictet and von Weizsacker, 1997). Ding, Granger, and Engle (1993), Ding, and Granger (1996) find the evidence that the actual sample volatility decays much slower than the exponential decay pattern as predicted by typical GARCH models. These stylized findings of heterogeneous volatility components and long memory volatility component motivated econometricians to develop component variance models that describe these findings. Most models distinguish the total conditional variance into short-run, long-run variance components and other components, such as seasonal variance component.

* Corresponding Author is Jang Hyung Cho and his email is cho.j@cob.sjsu.edu. Email of Ahmed F. Elshahat is aelshahat@bradley.edu.

1 For example, scalpers and day traders typically close positions before the end of the trading day. In contrast, position traders have longer interday horizons.

2 Exponential decay occurs when the decay rate of a function is proportional to the function’s current value. Geometric decay is discrete version of exponential decay. For example, \( \rho_t = a(\alpha + \beta)^{t-1} \) with \( \alpha + \beta \leq 1 \). Therefore, these two terms (exponential decay and geometric decay) are interchangeably used in the literature. For example, Andersen and Bollerslev (1997a) uses geometric decay to describe the decay pattern of the GARCH models.

3 Many studies have developed the variance (volatility) models that can accommodate the slow decay pattern of the variance process (long memory property). For example, the fractionally integrated GARCH (FIGARCH) as proposed by Baillie, Bollerslev and Mikkelsen (1996) and Bollerslev and Mikkelsen (1996). Comte and Renault (1998) further develop a fractionally integrated model that accounts for the stochastic nature of volatility. However, this focus of this study is both the short-run and the long-run variance components, not just long memory property. Hence, we do not emphasize the literature on the long memory property in this study.
In the two component GARCH models, the short-run conditional variance component captures transitory effects of volatility innovations, and the long-run variance component characterizes slower variations in the variance process associated with more permanent effects. Ding and Granger (1996) propose a two component GARCH model, where the first component is integrated variance component and the other is mean-reverting GARCH component. Maheu (2005) and Bauwens and Storti (2009) shows that Ding and Granger’s model can capture long-range variance dynamics with a slight modification of their models. Engle and Lee’s (1999) two component GARCH model does not restrict the long-run variance component to be integrated. Engle and Lee (1999) shows that allowing for long-run and short-run components greatly enhances a GARCH model’s ability to fit daily equity return dynamics. Engle and Rangel (2008) propose a spline-GARCH model that captures the permanent effect of news by an exponential quadratic spline curve. They found that the low frequency component is greater when the macroeconomic factors are more volatile.

Three component GARCH models describe the total conditional variance by short-run, long-run variance components and seasonal variance component (see Andersen and Bollerslev, 1997b; Engle, Sokalska and Chanda, 2006). These three component GARCH models are not compared with the model developed in this study because the long-run variance component is exogenously determined before these three component models are implemented. Furthermore, the focus of the three component models is the determination of the seasonal variance component.

The component GARCH model by Engle and Lee (1999) decomposes the total conditional variance into permanent and transitory variance components. This component GARCH model is widely used in academia. This study shows that the long-run variance component estimated by the Engle and Lee’s model is indistinguishable from the total variance in many cases. We identify the two main conditions of coefficients of the model under which the long-run variance component is not filtered from the total conditional variance from the Engle and Lee’s model. We suggest a modified component GARCH model to overcome the aforementioned mal-filtering problem which occurs in the Engle and Lee’s model. Our model is also different from those in Ding and Granger (1996) and Engle and Rangel (2008). Ding and Granger (1996) restrict the long-run variance component to be integrated. However, our model determines the long-run variance component such that the likelihood of sample observations is maximized. The long-run variance component in Engle and Rangel (2008) is assumed to be deterministic. However, the unobserved trend of long-run variance component is rather stochastically time-varying. We allow the long-run variance component to be stochastic. We employ spectral analysis to examine the long-run variance filtering performance of the modified component GARCH model with the benchmark of Engle and Lee’s model. The sample data are from the REIT (Real Estate Investment Trust) indexes of U.S. and Japan from year 2000 to 2009.

Results show that the estimated long-run variance component using the proposed modified component GARCH model is well distinguished from the total conditional variance, whereas virtually there is no difference between them from the Engle and Lee’s model for both the REITs on U.S. and Japan. Results from the power transfer function and the squared coherence spectrum show clearly that the mal-filtering problem which occurs in the results using the Engle and Lee’s model does not happen in the results from the modified component GARCH model. The estimated power transfer functions show that the coefficients of the Engle and Lee’s model allow the long-run variance component to contain the temporary variance component as well as permanent variance component, resulting in mal-filtering. The squared coherence (correlation as a function of frequency) of the unfiltered total variance component and the filtered long-run variance component from the Engle and Lee’s model is near 1.0 for all frequencies. For a good low-pass filtering (filtering only long-run variance component), the squared coherence should be low at low frequencies and close to one at high frequencies. The estimated squared coherence from the

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4 Literature uses other terms, such as temporary and permanent components (Engle and Lee, 1999) and high-frequency and low-frequency components (Engle and Rangel, 2008) for short-run and long-run volatility components, respectively. This study use the terms interchangeably.
modified component GARCH model show this pattern of squared coherence, representing good filtering performance.

We proceed in this study as follows: In section 2, we identify the conditions under which the Engle and Lee’s (1999) component GARCH model results in mal-filtering. Accordingly, we propose a modified component GARCH model. Section 3 reports the empirical results. We conclude in Section 4. The appendix shows the conditions of non-negativity and stationarity of the variance. We also derive the conditions on the coefficients of long-run variance component equation for a theoretically sound filtering performance in the appendix.

MODIFIED COMPONENT GARCH MODEL

Revisit to Engle and Lee’s (1999) model

Among the two component models, Engle and Lee’s (1999) two component GARCH model has been widely accepted in the literature. Speight, McMillan and Gwilym (2000) examine the decomposition of intraday conditional variance into the long-run and the short-run variance components using Engle and Lee’s model with the sample from FTSE 100. Hughes, Smith and Winters (2008) employ the Engle and Lee’s model to examine the U-shaped variance of T-bills. They include seasonal dummy variables in the equation of the long-run variance. The two component variance models are also applied in the option pricing literature (see Xu and Taylor, 1994; Bates, 2000; and Christoffersen, Jacobs, and Wang, 2006). Despite the significance of Engle and Lee’s two component GARCH model, we find the long-run variance component estimated by the Engle and Lee’s model is indistinguishable from the total variance in many cases. This section shows that the Engle and Lee component GARCH model does not predict the long-run variance component well under the two main conditions.

The Engle and Lee’s (1999) model is specified as follows:

\[ r_t = E[r_{t+1}] + e_t \]  
\[ e_t = \sqrt{h_t} v_t, \]  
\[ h_t = q_t + \alpha_1 (e_{t-1}^2 - q_{t-1}) + \beta_1 (h_{t-1} - q_{t-1}), \]  
\[ q_t = w + \phi (e_{t-1}^2 - h_{t-1}) + \rho q_{t-1}, \]

where \( v_t \sim N(1,0) \), \( h_t \) is the total conditional variance, \( q_t \) is the long-run variance component or the trend of the total conditional variance. The Engle and Lee’s model can be re-expressed as follows:

\[ h_t = \{w + (\rho - \alpha_1 - \beta_1)q_{t-1}\} + (\alpha_1 + \phi)e_{t-1}^2 + (\beta_1 - \phi)h_{t-1} \]  
\[ q_t = w + \phi (e_{t-1}^2 - h_{t-1}) + \rho q_{t-1}. \]

The above expression in (5) is obtained by substituting \( q_t \) into \( h_t \) equation. The conditions of non-negativity on the variance include \( \beta_1 \geq \phi \) and \( \alpha_1 \geq 0 \). Questions on filtering the long-run variance component arise in the cases where the difference between \( \beta_1 \) and \( \phi \) is small (\( \beta_1 \to \phi \) hereafter) and/or \( \alpha_1 \) is very small (\( \alpha_1 \to 0 \) hereafter). First, if \( \beta_1 \to \phi \), then \( h_t \) in (3) becomes as follows:

\[ h_t = w + (\alpha_1 + \phi)e_{t-1}^2 + \rho q_{t-1}. \]

\[ 5 \] Statistical packages also have built-in routines of the Engle and Lee’s (1999) component GARCH model. For example, Eviews.
By comparing the equations of $h_t$ in (6) and $q_t$ in (4), we can notice that there is very small difference between $h_t$ and $q_t$. This finding becomes clear by comparing the spectral densities of $h_t$ and $q_t$. The spectral density for (6) is

$$f_h = \rho^2 f_q + (\alpha_1 + \phi)^2 f_{e_{t-q}^2},$$

(7)

where $f_h$, $f_q$ and $f_{e_{t-q}^2}$ are the spectral densities of $h_t$, $q_t$ and $e_{t-q}^2$, respectively. Note that there is no cross spectrum between $q_t$ and $e_{t-q}^2$ because $e_{t-q}^2$ is net of $q_t$. Because $\rho \approx 1$ and $(\alpha_1 + \phi)^2 \approx 0$ in most case, we expect that $f_{e_{t-q}^2} \approx f_h$, or equivalently, $f_q/ f_h \approx 1$ for all frequencies. Recall that $q_t$ should contain only the low frequency component of $h_t$: $f_q/ f_h \approx 1$ only for low frequency and $f_q/ f_h \approx 0$ at high frequencies for a good low frequency component filtering.

Second, let’s consider the case where $\alpha_1 \rightarrow 0$. Under this condition, the total conditional variance equation in (5) becomes as follows:

$$h_t = w + \phi(e_{t-1}^2 - h_{t-1}) + \beta_1(h_{t-1} - q_{t-1}) + \rho q_{t-1}.$$  

(8)

Since $q_t = w + \phi(e_{t-1}^2 - h_{t-1}) + \rho q_{t-1}$, the equation in (8) is re-expressed by:

$$h_t = q_t + \beta_1(h_{t-1} - q_{t-1}).$$  

(9)

Or, equivalently,

$$(1 - \beta_1 L)h_t = (1 - \beta_1 L)q_t.$$  

(10)

The above equation in (10) shows that the process of $h_t$ and $q_t$ is identical under the condition that $\alpha_1 \rightarrow 0$. From the perspective of spectral analysis, the spectral density for the equation in (10) is

$$f_h = (1 + \rho^2 - 2\beta_1 \cos(\omega))f_q = (1 + \rho^2 - 2\beta_1 \cos(\omega))f_q.$$  

(11)

Or, equivalently,

$$f_h = f_q.$$  

(12)

Therefore, if $\alpha_1 \rightarrow 0$, then $f_q \rightarrow f_h$ at all frequencies. As a result, $f_q/ f_h \rightarrow 1$. This result means that the equation for $q_t$ in (4) does not filter the low frequency component at all. It is obvious that the permanent variance component is not distinguishable from the total conditional variance under the combination of the above two cases ($\alpha_1 \rightarrow 0$ and $\beta_1 \rightarrow \phi$) i.e., $h_t = q_t$.

**MC–GARCH model**

Given the issues in the Engle and Lee’s (1999) component GARCH model, we propose a modified component GARCH model which overcomes the mal-filtering problem and offers better filtering performance. The better filtering performance is achieved by increasing the low frequency component and by decreasing the high frequency component in $q_t$. We modify the Engle and Lee’s model by redefining the innovation in long-run variance component $q_t$. Therefore, we begin by the definition of innovation.

Brown, Durbin and Evans (1975) note that the innovation at time $t$ is defined as the difference between the observation at time $t$ and its best linear predictor given all the observations up to and including time $t-1$ in a time series. The $q_t$ is the low frequency component of $h_t$ as the $h_t$ is that of $e_{t-1}^2$. Also, as $e_{t-1}^2$ is the moving average term (MA) of $h_t$, the MA of $q_t$ should be of $h_t$. Put differently, because $q_t$ is the trend of $h_t$, the innovation of $q_t$ is from $h_t$. Therefore, on the ground of the definition of innovation
in Brown et al. (1975), the innovation in the long-run variance process should be $h_{t-1} - q_{t-1}$ rather than $e_{t-1}^2 - h_{t-1}$. Notice that the innovation of the long-run variance component $q_t$ in the Engle and Lee’s model is $e_{t-1}^2 - h_{t-1}$ as shown in (4). Consequently, we propose the following modified component GARCH (MC-GARCH hereafter) model:

$$\begin{align*}
n_t &= E[r_t] + e_t, \\
e_t &= \sqrt{h_t} v_t, \\
h_t &= q_t + \alpha_t (e_{t-1}^2 - q_{t-1}) + \beta_t (h_{t-1} - q_{t-1}), \\
q_t &= w + \phi (h_{t-1} - q_{t-1}) + \rho q_{t-1},
\end{align*}$$

\(13\)

In the remainder of this section we use the Engle and Lee’s (1999) model as a benchmark to show how the proposed model offers superior filtering in forecasting the long-run variance component. The low frequency variance component $q_t$ in (13) as follows:

$$q_t = w + \phi h_{t-1} + (\rho - \phi) q_{t-1},$$

\(14\)

The spectral density of $q_t$ (denoted by $f_q$) in terms of the spectral density of $h_t$ (denoted by $f_h$) is obtained as follows:

$$f_q(\omega) = T(\omega|\phi, \rho)f_h(\omega),$$

\(15\)

where

$$T(\omega|\phi, \rho) = \frac{\phi^2}{1 + (\rho - \phi)^2 - 2(\rho - \phi)\cos(\omega)}.$$  

\(16\)

The coefficient $T(\omega|\phi, \rho)$ in (16) is the power transfer function (PTF) of the MC-GARCH model, which represents how much of the variance of $h_t$ is maintained in $q_t$ at the frequency $\omega$. The bounds of PTF is that $0 \leq T \leq 1$. This PTF of MC-GARCH is illustrated in Figure 1. Figure 1 shows the $q_t$ in (14) allows to pass only the low frequency component and block the high frequency component of $h_t$. Moreover, the low frequency filtering of $q_t$ in (13) is not vulnerable to the behaviors of $\alpha_t$ and $\beta_t$ at all i.e., the PTF of the MC-GARCH is not a function of $\alpha_t$ and $\beta_t$. Refer to Appendix for the conditions of non-negativity and stationarity of the variance. Appendix also provides the conditions for filtering low frequency component.

**EMPIRICAL EXAMINATION**

**Results of estimation**

We examine the MC-GARCH model using daily datasets from the REIT indexes of U.S. and Japan from year 2000 to 2009. The returns on REIT are computed by $100\ln(P_t/P_{t-1})$, where $\ln$ is the natural logarithm, and $P_t$ is the level of time series.

Figure 2 shows the estimated total conditional variance and the predicted long-run variance components using the Engle and Lee’s (1999) model and MC-GARCH model. Results show that the estimated long-run variance component using the MC-GARCH model is well distinguished from the total conditional variance. However, virtually there is no difference between the total conditional variance and

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\(^6\) See chapter 4 in Fuller (1995) for details on filtering.
the long-run variance component from the Engle and Lee’s model for both the REITs on U.S. and Japan. The two series share almost same values. As mentioned earlier, these results of undistinguishable long-run variance component from the total conditional variance occur if the difference between $\beta_1$ and $\varphi_1$ is small and/or the value $\alpha_1$ converges to zero. The estimation results in Table 2 confirm these expectations. The value of $\alpha_1$ for REIT of U.S. is very small in size (0.069) under which condition the filtered long-run variance component becomes indistinguishable from the total conditional variance i.e., $q_t \rightarrow h_t$. The results for REIT of Japan using the Engle and Lee’s model shows that $\beta_1 = \varphi = 0.124$. If $\beta_1 = \varphi$ in the Engle and Lee component GARCH model, then we have seen that the conditional variance equation is hardly distinguished from the long-run variance component. The estimation results using the MC-GARCH model show that the estimated values of $\beta_1$ and $\varphi$ are sufficiently different 0.814 and 0.029 for REIT of U.S. and 0.677 and 0.032 for REIT of Japan, respectively. Also, the values of $\alpha_1$ are sufficiently larger than zero for both series. Therefore, the estimation results show that the long-run variance component is well filtered in the MC-GARCH model.

Results of filtering performance

We report results of the low frequency variance filtering performance more formally by the estimated values of power transfer function and the squared coherence in Figure 3 and 4, respectively. The coherence is a measure of the correlation between two series as a function of the frequency (or equivalently, period). Hence, for the two identical time series, the coherence is one. A good low-pass filtering procedure removes the high frequency component from the unfiltered series. As a result, the filtered series contains only the low frequency component, and there is a low correlation between the unfiltered and filtered series only at the high frequencies for the good low-pass filtering procedure. Therefore if the coherence is statistically different from 1.0 at the high frequencies, then the filtering process has successfully removed variance from the series at these low periods. If the coherences between the unfiltered and filtered series are indistinguishable from 1.0 at high frequencies (low periods), then a maladjustment of the high frequency (low periods) variance component exists.

Results in Figure 3 show that the theoretic power transfer function (PTF) derived in (16) perfectly fits the estimated PTF for both the REITs of U.S. and Japan. Recall that the power transfer function represents how much of the variance of $h_t$ is maintained in $q_t$ at the frequency $\omega$. In order for $q_t$ to be a long-run variance component, $q_t$ should include only the low frequency variance component. The high frequency variance component represents the temporary variance component. Results show that the estimated values of PTF for the REITs of U.S. from the Engle and Lee’s model are not sufficiently small at high frequencies i.e., the long-run variance component from the Engle and Lee’s model includes the high frequency variance component. The estimated PTF for the REITs of Japan using the Engle and Lee’s model in Figure 3 maintain high values over both the high and low frequencies. These results for the REITs of Japan using Engle and Lee’s model represents that the long-run variance component include most of the high frequency variance component (temporary variance component).

We report the results on how well the models filter the low frequency component with the squared coherence in Figure 4. Results confirm that the estimated values of correlation (coherence) between the unfiltered total conditional variance and the filtered long-run variance component from the Engle and Lee’s model is near 1.0 at all frequencies. These results mean that the low frequency component is not properly filtered from the total variance using the Engle and Lee’s model. Meanwhile, the correlation between the unfiltered total conditional variance and the filtered long-run variance component from the MC-GARCH model is near 1.0 only at low frequencies (high periods). These results clearly represent that the filtered variance has correlation with the unfiltered variance only for the low frequencies.

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7 Period is the reciprocal of frequency.
CONCLUSION

Component GARCH models are important because the market volatility aggregates heterogeneous volatility components, where the heterogeneous volatilities represent differing information. Among the component models, Engle and Lee’s (1999) two component GARCH model widely used in the literature. Many empirical studies on long-run and transitory variances use the Engle and Lee’s model for stocks and options.

However, despite the significance of Engle and Lee’s two component GARCH model, we find the long run variance component estimated by the Engle and Lee’s model is indistinguishable from the total variance in many cases. We identify the two main conditions of coefficients of the model under which the long-run variance component is not filtered from the total conditional variance from the Engle and Lee’s model. Accordingly, we develop a modified component (MC) GARCH model to overcome the aforementioned mal-filtering problem which occurs in the Engle and Lee’s model. We also show that our model is also different from those in Ding and Granger (1996) and Engle and Rangel (2008). We employ spectral analysis to examine the long-run variance filtering performance of the MC-GARCH model with the benchmark of Engle and Lee’s model. The sample data are from the REIT indexes of U.S. and Japan from year 2000 to 2009.

Results show that the estimated long-run variance component using the MC-GARCH model is well distinguished from the total conditional variance, whereas virtually there is no difference between them from the Engle and Lee’s model for both the REITs on U.S. and Japan. Results from the power transfer function and the squared coherence spectrum show clearly that the mal-filtering problem which occurs in the results using the Engle and Lee’s model does not happen in the MC-GARCH model. The estimated power transfer functions show that the coefficients of the Engle and Lee’s model allow the long-run variance component to contain the temporary variance component, resulting in mal-filtering. The squared coherence (correlation as a function of frequency) of the unfiltered total variance component and the filtered variance component from the Engle and Lee’s model is near 1.0 for all frequencies. For a good low-pass filtering (filtering only long-run variance component), the squared coherence should be low at low frequencies and close to one at high frequencies. The estimated squared coherence from the MC-GARCH model show this pattern of squared coherence, representing good filtering performance.
The above graphs show the theoretic power transfer function in (16).

\[ T(\omega|\phi, \rho) = \frac{\phi^2}{1 + (\rho - \phi)^2 - 2(\rho - \phi)\cos(\omega)} \]

The power transfer function represents how much of the unfiltered total variance of \( h \) is filtered into \( q \) at the frequency \( \omega \). Frequency is angular frequency which is \( 2\pi/N \), where \( N \) is the number of total observations.
Figure 2 shows the unfiltered total conditional variance and the filtered long-run variance components from the Engle and Lee (1999) component GARCH and the MC-GARCH models.
Figure 2: The Estimated Total Conditional Variance and Long-Run Variance (Continued)
Figure 3 shows the estimated and theoretic power transfer function in (16) the estimated long-run variance components from the Engle and Lee (1999) component GARCH and the MC-GARCH models using the returns on REITs of U.S. and Japan.

\[
T(\omega|\phi, \rho) = \frac{\phi^2}{1 + (\rho - \phi)^2 - 2(\rho - \phi)\cos(\omega)}
\]

The power transfer function represents how much of the unfiltered total variance of \( h \), is filtered into \( q \), at the frequency \( \omega \). Frequency is angular frequency which is \( 2\pi/N \), where \( N \) is the number of total observations. Therefore, good filtering procedure should exhibit the power transfer functions with high values at the low frequencies and the low values (preferably, zero values) at the high frequencies.
Figure 4 shows the estimated squared coherence. The coherence is a measure of the correlation between two series as a function of the frequency (equivalently, period). Hence, for the two identical time series, the coherence is one. If the coherence is statistically different from 1.0 at the high frequencies, then the filtering process has successfully removed variance from the series at these high frequencies. If the coherences between the unfiltered and filtered series are indistinguishable from 1.0 at high frequencies (low periods), then a maladjustment of the high frequency (low periods) variance component exists.
Table 1: The Bounds of the Values of Phi (φ) to Filter the Low Frequency Variance Component

Table 1 shows the values of φ for the 0% to 5% of the power transfer function in (16) at the cutoff frequency of $\omega_c = 2\pi/30 \approx 0.21$. The value of φ is computed by

$$\phi = \frac{T(\rho - \cos(\omega)) + \sqrt{T(\rho - \cos(\omega))^2 + (1 - T)(1 - \cos^2(\omega))}}{1 - T}.$$ 

In the table “T” stands for the value of power transfer function. If T=0.05, then the MC-GARCH allows the long-run variance component to contain high frequency component (cycles with periods less than 30 days) by 5%.

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Table 2: Estimation Results of Component GARCH Models

Table 2 provides the estimation results from the Engle and Lee (1999) component GARCH and the MC-GARCH models. Below are the variance equations of the component GARCH models.

The total conditional variance:
\[ h_t = q_t + \alpha_1 (e_{t-1}^2 - q_{t-1}) + \beta_1 (h_{t-1} - q_{t-1}) \]

Long-run variance component in Engle and Lee (1999) model:
\[ q_t = \omega + \phi (e_{t-1}^2 - h_{t-1}) + \rho q_{t-1} \]

Long-run variance component in MC-GARCH model:
\[ q_t = \omega + \phi (h_{t-1} - q_{t-1}) + \rho q_{t-1} \]

***, ** and * indicate statistical significance at 1, 5 and 10 percent, respectively.

Panel A: Returns on REIT index of U.S.

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<td>Rho</td>
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Panel B: Returns on REIT index of Japan

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APPENDIX
A.1. Conditions of stationarity and non-negativity

If a time series is non-stationary, then the coefficients of a linear model for the time series cannot be determined. Furthermore, the variance cannot be negative. Hence, we find the conditions of stationary and non-negativity of the variance processes. The low frequency variance component \( q_t \) of the MC-GARCH model can be re-expressed as follows.

\[
q_t = \frac{w}{1-(\rho - \phi)} + \frac{\phi}{1-(\rho - \phi)L} h_{t-1},
\]

(A1)

If the \( q_t \) is substituted into \( h_t \), then we can show that the MC-GARCH model is equivalent to the regular GARCH \((1,\infty)\) as shown below.

\[
h_t = w \left[ 1 + \frac{\rho - \alpha_1 - \beta_1 - \phi}{1 - \rho + \phi} \right] + \alpha_1 e_{t-1}^2 + (\beta_1 + \phi) h_{t-1} + \frac{(\rho - \alpha_1 - \beta_1 - \phi)\phi}{1-(\rho - \phi)L} h_{t-2}
\]

(A2)

Hence, it is easy to see that the non-negative of the conditional variance \( h_t \) is insured with the following conditions:

\[
\alpha_1 \geq 0, \beta_1 \geq 0, w \geq 0, \phi \geq 0, \rho \geq 0, \rho \geq \alpha_1 + \beta_1 + \phi.
\]

(A3)

Stationarity of \( h_t \) will be satisfied if the sum of coefficients in (A2) is less than unity. Specifically,

\[
\alpha_1 + \beta_1 + \phi + \frac{(\rho - \alpha_1 - \beta_1 - \phi)\phi}{1-(\rho - \phi)} < 1
\]

(A4)

The above condition in (A4) is reduced to:

\[
\alpha_1 + \beta_1 < 1.
\]

(A5)

Stationarity of \( q_t \) is satisfied if

\[
\rho < 1.
\]

(A6)

A.2. Conditions of filtering the long-run variance component

Although the values of \( \phi \) and \( \beta_1 \) in (A2) are uniquely determined by the Maximum likelihood (ML) method, the value of \( \phi \) determined by ML can be too large to filter the low frequency component. Therefore, to achieve the better low frequency variance filtering performance, we need to have conditions on \( \phi \).

Typical low-pass filtering procedure requires researchers’ judgment for the cut-off frequency, for example, the Baxter and King’s (1999) filtering procedure. Baxter and King’s filtering procedure can accommodate any cut-off frequencies which are determined by a researcher’s needs. However, the way to filter the low frequency component from the total conditional variance in the component GARCH model is limited to the restrictions on the coefficients in \( h_t \) and \( q_t \). Therefore, we need restrictions on the coefficients for the purpose of filtering the low frequency component of the conditional variance.

The process \( q_t \) in (4) can be seen as filtered series using original time series \( h_t \), where the parameters \( w, \phi \) and \( \rho \) are filters for \( q_t \). The spectral density of \( q_t \) (denoted by \( f_q \)) in terms of the spectral density of \( h_t \) (denoted by \( f_h \)) is obtained as shown in equation (15) and (16). As mentioned earlier, the \( T(\omega|\phi,\rho) \) in (16) is the power transfer function, which represents how much of the variance of \( h_t \) is maintained in \( q_t \) at the frequency \( \omega \). We derive the conditions on filters \( \phi \) and \( \rho \) for filtering low frequency variance component using the power transfer function \( T(\omega|\phi,\rho) \).

The ideal low-pass filtering procedure requires the researcher’s choice of a cut-off frequency. Given a cutoff frequency \( \omega_c \), we should be find the conditions on filters \( \phi \) and \( \rho \) to maximize the \( T(\omega|\phi,\rho) \) for \( \omega < \omega_c \) and minimize \( T(\omega|\phi,\rho) \) for \( \omega \geq \omega_c \). In other words, we have to maximize the low frequency variance component and minimize the high frequency variance component, where the high and low frequencies are distinguished by the cutoff frequency point. For example, we may consider the cycles with longer than 30 days. The frequency of the cycle with longer than 30 days is 1/30 as the temporal
frequency and $2\pi/30$ as the angular frequency. Hence, the low frequency variance component is the variance series which is composed of the cycles with frequencies less than $\omega_c = 2\pi/30 \approx 0.21$.

The power transfer function $T(\omega|\varphi, \rho)$ is a continuously decreasing function in frequency $\omega$ given particular filters of $\varphi$ and $\rho$ as shown in Figure 1. The maximization of low frequency component and minimization of high frequency component is hardly achieved by any clear-cut mathematical analysis. Because the power transfer function is a continuous function in frequency, we encounter the problem of more inclusion of high frequency component when we try to increase low frequency component. However, if we include the cut-off frequency component (and therefore the high frequency component) too little then the low frequency part also becomes small. If the value of power transfer function at the cut-off frequency increases, then the performance of filtering substantially decreases. We find the values of $\varphi$ are 0.02 and 0.05, as the minimum and the maximum values for the filtering purpose under the cut-off frequency of $\omega_c = 2\pi/30$, which are robust to the values of $\rho$ from 0.9 to 1.0. In most cases, the value of $\rho$ is higher than 0.9.\(^8\) At $\varphi = 0.02$ and 0.05, the $q_t$ allows the high frequency component to pass by 1% and 5%, respectively. The value of $\varphi$ is determined by the following function.

$$\varphi = \frac{T(\rho - \cos(\omega)) + \sqrt{T(\rho - \cos(\omega))^2 + (1 - T)[1 - \cos^2(\omega)]}}{1 - T}$$

for $T \leq 1$, \hspace{1cm} (A7)

and $\varphi = 0$ for $T = 0$. The value inside the squared root in (A7) is always positive or zero.

\(^8\) The rigorous determination of the bounds of $\varphi$ can be determined by the following two steps. The first step determines the cutoff frequency $\omega_c$ and the minimum and maximum values of transfer function at $\omega_c$. In the second step, the minimum and maximum bounds of $\varphi$ for different values of $\rho$ are determined by the power transfer function. However, practically, the bounds of $\varphi$ can be simply set by 0.021 and 0.048 for $\rho$ from 0.95 to 1.0 since the value of $\rho$ should be larger than the sum of $\alpha_1$, $\beta_1$ and $\varphi$, where the sum of $\alpha_1$, $\beta_1$ and $\varphi$ has the values near 1.0 in most cases.
REFERENCES


